percentage of zeroes in the Jacobian (i.e., the sparsity) vs the number of elements. The $x(T)$ column shows that the 32-element case has almost converged on the exact solution. (Recall that the exact solution is $-17/39 = -0.43590$.) Note further that the approximate $x(T)$ is not an upper bound of the exact value, which is common in mixed formulations. The third column of Table 1 gives the elapsed computer time for five iterations. It is easily seen that there is a modest increase in computer time with an increase in the number of elements. Note that in some cases a converged answer is found in five or fewer iterations. This is because the answers obtained from a small number of elements (say two or four) may be interpolated to generate initial guesses for a higher number of elements. Thus, it is possible to solve a 16- or 32-element case in about 1.5 s. Finally, the extremely sparse structure of the Jacobian is demonstrated in the last column. This strongly encourages the use of a smart sparse matrix solver such as MA28. This subroutine leads to quicker solutions and tremendous savings in memory allocation since only the nonzeroes of the Jacobian need be stored.

Results for the control u are shown in Fig. 1 for two, four, and eight elements and the exact solution. Note that although the two-element case does not define the constraint boundaries very accurately, it is accurate enough to generate guesses for the four-element case. Thus, in a problem with many constrained and unconstrained arcs, a small number of elements could still be used to generate guesses for a higher number of elements. Also, it is interesting to note that as few as four elements have essentially converged on the exact solution.

Conclusions

In this Note, it has been shown that the weak Hamiltonian finite element formulation is amenable to the solution of optimal control problems with inequality constraints that are functions of both state and control variables. Difficult problems may be treated due to the ease with which algebraic equations can be generated before having to specify the problem. These algebraic equations yield very accurate solutions. Furthermore, due to the sparse structure of the resulting Jacobian, computer solutions may be obtained quickly when the sparsity is exploited.

Acknowledgments

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Equivalence of Two Classes of Dual-Spin Spacecraft Spinup Problems

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Introduction

SPINUP problems for dual-spin spacecraft have been investigated by numerous authors.[1-6] Usually researchers focus on a particular model; for example, Gebman and Mingori[1] and Guelman[4] specifically restricted their attention to the attitude recovery problem for prolate spacecraft. Others have dealt with oblate, prolate, and intermediate spacecraft separately, for example, Hubert.[3] In this Note we give a symmetry transformation for axial dual spinners that allows prolate spacecraft to be treated as oblate. This transformation is quite simple and should prove useful in future investigations of spinup problems.

We begin, of course, with the equations of motion and definitions for the terms oblate, prolate, and intermediate. Following a brief discussion of the transformation, we give an example relating two spacecraft. We then discuss the significance of this result and suggest some possible applications.

Equations of Motion

In the first approximation, a dual-spin spacecraft is usually modeled as two rigid bodies: a platform $\mathcal{P}$ and a rotor $\mathcal{R}$, connected by a shaft that allows relative rotation between them. We denote the system as $\mathcal{P}+\mathcal{R}$ (see Fig. 1). In this work, $\mathcal{P}$ is asymmetric, whereas $\mathcal{R}$ is axisymmetric about the axis of relative rotation e_1, which is a principal axis of $\mathcal{P}+\mathcal{R}$. An internal motor is used to provide an equal and opposite torque to each body along the connecting shaft.

The differential equations for the system $\mathcal{P}+\mathcal{R}$ with no external torque are[7]

$$\dot{h}_1 = \frac{I_2 - I_3}{I_2 I_3} h_2 h_3 \tag{1}$$

$$\dot{h}_2 = \left(\frac{I_3 - I_p}{I_3 I_p} h_1 - \frac{h_a}{I_p}\right) h_3 \tag{2}$$

$$\dot{h}_3 = \left(\frac{I_p - I_2}{I_2 I_p} h_1 + \frac{h_a}{I_p}\right) h_2 \tag{3}$$

$$\dot{h}_a = g_a \tag{4}$$

where

$h_a = I_s(\omega_s + \omega_1)$ = angular momentum of $\mathcal{R}$ about e_1
$h_1 = I_1\omega_1 + I_s\omega_s$ = angular momentum of $\mathcal{P}+\mathcal{R}$ about e_1
$h_i = I_i\omega_i$ = angular momentum of $\mathcal{P}+\mathcal{R}$ about e_i $(i = 1,2,3)$
I_i = moment of inertia of $\mathcal{P}+\mathcal{R}$ about e_i $(i = 1,2,3)$
I_s = moment of inertia of $\mathcal{R}$ about e_1
$I_p = I_1 - I_s$ = moment of inertia of $\mathcal{P}$ about e_1
ω_i = angular velocity of $\mathcal{P}$ about e_i $(i = 1,2,3)$
ω_s = angular velocity of $\mathcal{R}$ about e_1 relative to $\mathcal{P}$
g_a = torque applied by $\mathcal{P}$ on $\mathcal{R}$ about e_1
e_i = principal axes of $\mathcal{P}+\mathcal{R}$ $(i = 1,2,3)$

It is evident from Eqs. (1–3) that the magnitude of I_p relative to I_2 and I_3 plays an important role. Indeed, this prompts our

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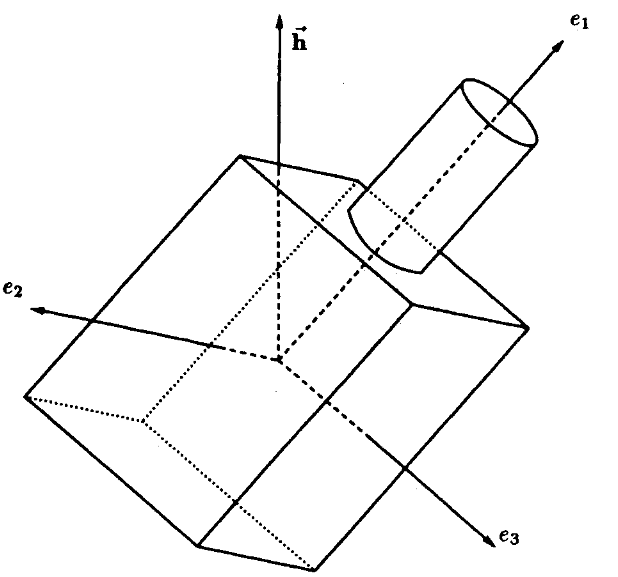

Fig. 1 Axial dual-spin spacecraft $\mathcal{P}+\mathcal{R}$: $\mathcal{P}$ is rigid platform; $\mathcal{R}$ is rigid axisymmetric rotor; e_i, $i=1,2,3$ are principal axes of $\mathcal{P}+\mathcal{R}$. The total angular momentum vector h is constant in direction and magnitude.

choice of the following definitions, in which we assume without loss of generality that $I_2 > I_3$:

A spacecraft is oblate if $I_p > I_2 > I_3$.

A spacecraft is prolate if $I_2 > I_3 > I_p$.

A spacecraft is intermediate if $I_2 > I_p > I_3$.

Another common definition for prolateness is $I_2 > I_3 > I_1$ (Gebman and Mingori[1]). We note that since $I_1 = I_p + I_s > I_p$, this definition also satisfies our criterion.

Since there are no external moments, angular momentum is conserved and a first integral of the motion is

$$h_1^2 + h_2^2 + h_3^2 = h^2 = \text{const} \tag{5}$$

which defines a sphere of radius h in $\boldsymbol{R}^3$. If $g_a = 0$, Eq. (4) implies $h_a = h_a^* = \text{const}$, and there are either two, four, or six equilibrium points on the sphere depending on the value of h_a^* (see Hughes[7]). When $g_a \neq 0$, however, there are exactly two equilibrium points on the sphere: one at the "north pole," $(h_1,h_2,h_3)=(h,0,0)$, and one at the "south pole" $(-h,0,0)$. For most spacecraft, the desired operating condition is the equilibrium point at the north pole with $h_a = h_a^* = \text{const}$. At this equilibrium, the platform rotates about the rotor axis with angular velocity $\omega_1 = (h - h_a^*)/I_p$. Of particular usefulness is the case where $h_a^* = h$, which gives $\omega_1 = 0$, the dual-spin condition. During the spinup process, the behavior of solutions to Eqs. (1-4) near this equilibrium is markedly different for the three different spacecraft configurations.[6] The symmetry transformation we describe next shows that the north pole equilibrium for a prolate spacecraft is identical to the south pole equilibrium for an equivalent oblate spacecraft. This means that, although the local dynamics are different for the different configurations, the global dynamics are the same for oblate and prolate spacecraft.

Symmetry Transformation

It is easy to verify by direct substitution the following proposition.

Proposition 1

Equations (1-4) are invariant under the transformation

$$(h_1,h_2,h_3,I_2,I_3) \mapsto (-h_1,h_3,-h_2,\hat{I}_3,\hat{I}_2) \tag{6}$$

where

$$\hat{I}_2 = \frac{I_p I_2}{2I_2 - I_p} \tag{7}$$

$$\hat{I}_3 = \frac{I_p I_3}{2I_3 - I_p} \tag{8}$$

are the transformed moments of inertia.

By itself, this transformation is merely interesting; the next proposition tells us how useful it is.

Proposition 2

Given a prolate spacecraft, $I_p < \min\{I_2,I_3\}$, application of Eqs. (7) and (8) gives transformed moments of inertia that satisfy $I_p > \max\{\hat{I}_2,\hat{I}_3\}$. Furthermore, if I_p, I_2, and I_3 satisfy the "triangle inequalities" for moments of inertia, then so do I_p, $\hat{I}_2$ and $\hat{I}_3$. Conversely, $I_p > \max\{I_2,I_3\}$ implies $I_p < \min\{\hat{I}_2,\hat{I}_3\}$; however, when used in this sense, the transformed moments of inertia will not necessarily satisfy the triangle inequalities.

Thus the transformation maps a prolate spacecraft to an equivalent oblate spacecraft. We point out that the h_i part of the transformation amounts to inversion of the momentum sphere $(h_1 \mapsto -h_1)$, along with a rotation $(h_2,h_3 \mapsto h_3,-h_2)$. It is important to note that the angular velocities do not transform in an equivalent way. The usual dual-spin condition $(h_1,h_2,h_3,h_a)=(h,0,0,h)$ gets mapped to $(-h,0,0,h)$, and the corresponding platform angular velocity becomes $\omega_1 = -2h/I_p$, not $\omega_1 = 0$. Hence one must invert the transformation if the angular velocities are required.

Example

To illustrate the use of the transformation, we choose a prolate spacecraft with moments of inertia given by

$$(I_p,I_2,I_3) = (30,60,40) \tag{9}$$

Note that this spacecraft could have e_1 as its major, minor, or intermediate axis, since we have not specified the axial moment of inertia of the rotor I_s. Specifically, since $I_1 = I_p + I_s$, a large rotor $(I_s > 30)$ would make this a major axis spacecraft, whereas a small rotor $(I_s < 10)$ would give a minor axis spacecraft, and an intermediate value for I_s $(10 < I_s < 30)$ would give an intermediate axis spacecraft.

Applying the transformation of Eqs. (7) and (8) to the moments of inertia in Eq. (9) gives transformed moments of inertia

$$(I_p,\hat{I}_3,\hat{I}_2) = (30,24,20) \tag{10}$$

which satisfy our definition for an oblate spacecraft. The specific expressions of Eqs. (1-4) for the two spacecraft are given in Table 1. [Note that Eqs. (1) and (4) are unaffected by the transformation.] The equations for the prolate spacecraft are transformed to those for the equivalent oblate spacecraft by applying the h_i part of Eq. (6), namely $(h_1,h_2,h_3) \mapsto (-h_1,h_3,-h_2)$. Thus the dynamics of either the prolate spacecraft or the equivalent oblate spacecraft may be studied using either of the two sets of equations in Table 1.

For example, a typical spinup maneuver for a prolate spacecraft with e_2 as the major axis is the flat spin recovery,[1] where the initial conditions are $(h_1,h_2,h_3,h_a)=(0,\pm h,0,0)$, with $g_a = 0$. Then the motor is turned on $(g_a \neq 0)$ to begin spinup, with $(h_1,h_2,h_3,h_a)=(h,0,0,h)$ as the desired final state. To simulate this maneuver using the equations for the oblate spacecraft in Table 1, the *transformed* initial conditions would

Table 1 Equations (1-4) for equivalent spacecraft

Prolate $(I_p,I_2,I_3)=(30,60,40)$	Oblate $(I_p,I_2,I_3)=(30,24,20)$	Eq.
$\dot{h}_1 = h_2h_3/120$	$\dot{h}_1 = h_2h_3/120$	(1)
$\dot{h}_2 = (h_1/120 - h_a/30)h_3$	$\dot{h}_2 = -(h_1/60 + h_a/30)h_3$	(2)
$\dot{h}_3 = -(h_1/60 - h_a/30)h_2$	$\dot{h}_3 = (h_1/120 + h_a/30)h_2$	(3)
$\dot{h}_a = g_a$	$\dot{h}_a = g_a$	(4)

be $(h_1, h_2, h_3, h_a) = (0, 0, \mp h, 0)$, and the desired final state would be $(h_1, h_2, h_3, h_a) = (-h, 0, 0, h)$.

Significance of the Transformation

Unifying transformations of this kind are always of academic interest and often of practical importance as well. This result is interesting because it helps us to understand the similarities and differences between the different spinup problems, but it is also very useful. An immediate application is in developing a graphics program to generate three-dimensional views of the momentum sphere. Since the sphere associated with a prolate spacecraft may be obtained by inverting and rotating the sphere for an equivalent oblate spacecraft, it is unnecessary to deal specifically with the prolate case, resulting in a simpler program design.

Another application is in the development of a perturbation analysis of Eqs. (1-4) for small g_a. Since the $g_a = 0$ solution involves Jacobi's elliptic functions,[8] a perturbation treatment can be quite complicated.[6] The oblate/prolate equivalence reduces the number of cases that must be considered.

Conclusions

The distinction between oblate and prolate dual spinners has been made by most researchers of these problems. In this Note we presented a simple transformation that relates the global dynamics of the two types of spacecraft. We also pointed out the significance of the result and offered some sample applications.

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Stability of a Rate Gyro

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Nomenclature

A, C	= transverse and axial moments of inertia of rotor
A_g, B_g, C_g	= principal moments of inertia of gimbal
C_d, K	= damping coefficient and spring constant, respectively
$C_d\dot{\theta}, K\theta$	= terms of first degree of Taylor series expansions of $f_1(\dot{\theta})$ and $f_2(\theta)$, respectively
D_1, D_2, D_3, D_4	= $C_d/n(A+A_g)$, $K/n^2(A+A_g)$, $C/n(A+A_g)$, $(A+B_g-C_g)/n^2(A+A_g)$, respectively
$f_1(\dot{\theta}), f_2(\theta)$	= nonlinear torsional damping torque and spring torque, respectively
$m_1, m_2, \lambda_1, \lambda_2, \lambda_3$	= undetermined positive constants
n	= rated angular velocity of rotor, $\dot{\psi} - \omega_Y \sin\theta + \omega_Z \cos\theta$
OX, OY, OZ	= orthogonal axes fixed with platform, where OY is input axis, OX is output axis of gimbal and always coincides with Ox
Ox, Oy, Oz	= principal axes of inertia of rotor and gimbal, which coincide with OX, OY, OZ, respectively, in the equilibrium position
α, β	= θ, θ', respectively
θ	= output deflection angle between rotor axis Oz and OZ
θ', θ''	= $d\theta/d\tau$, $d^2\theta/d\tau^2$, respectively
τ	= dimensionless time, nt
$\dot{\psi}$	= spin angular velocity of rotor
$\omega_X, \omega_Y, \omega_Z$	= components of angular velocity of rotor along OX, OY, OZ, respectively, where ω_Y is the input

Introduction

FOR many practical applications, it is necessary to measure the angular velocity of a given vehicle. The angular velocity about an axis normal to a given platform can be recorded by means of a single-gimbal gyro whose spin axis is normal to the axis of motion. Such a gyro is referred to as a rate gyro.

Using Lagrange's equation, we can derive the following equation of motion for the output deflection angle θ of a rate gyro[1] (see Fig. 1):

$$(A+A_g)\ddot{\theta} + f_1(\dot{\theta}) + f_2(\theta) + Cn(\omega_Y \cos\theta + \omega_Z \sin\theta)$$
$$+ (A+B_g-C_g)(\omega_Y \cos\theta + \omega_Z \sin\theta)(\omega_Y \sin\theta - \omega_Z \cos\theta)$$
$$= -(A+A_g)\dot{\omega}_X \qquad (1)$$

where

$$Cn = C(\dot{\psi} - \omega_Y \sin\theta + \omega_Z \cos\theta) = \text{const} \qquad (2)$$

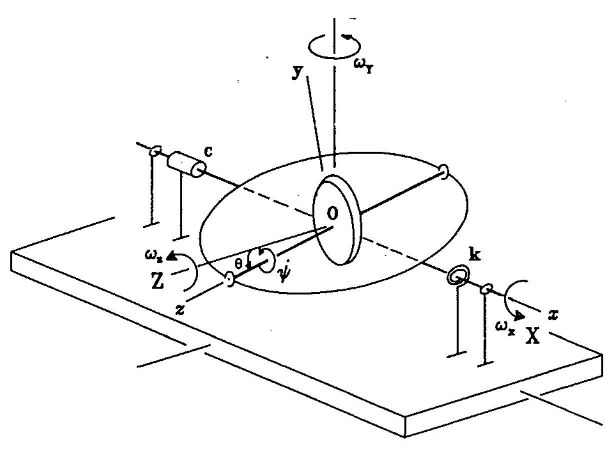

Fig. 1 Rate gyro.

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